MATH 121A Prep: Diagonalization

1. Diagonalize the matrix $A = \begin{bmatrix} -2 & 0 \\ 10 & 3 \end{bmatrix}$.

Solution: First we need the eigenvalues by solving the polynomial

$$\det(A - \lambda I_2) = \begin{vmatrix} -2-\lambda & 0 \\ 10 & 3-\lambda \end{vmatrix} = (-2-\lambda)(3-\lambda)$$

so the eigenvalues are -2 and 3.

$\lambda = -2$

$$(A - \lambda I_2)\vec{v} = \vec{0} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 10 & 5 & 0 \end{bmatrix} \xrightarrow{R2=1/5R2} \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

so x_1 is free and $x_2 = -2x_1$ so $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector.

$\lambda = 3$

$$(A - \lambda I_2)\vec{v} = \vec{0} \rightarrow \begin{bmatrix} -5 & 0 & 0 \\ 10 & 0 & 0 \end{bmatrix} \xrightarrow[R2=R2-2R1]{R1=-1/5R1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so $x_1 = 0$ x_2 is free so $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector.

Therefore $A = PDP^{-1}$ where

$$D = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

2. Use your answer from Question 1 to compute A^3.

Solution:

$$\begin{aligned} A^3 &= PD^3P^{-1} \\ &= \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -8 & 0 \\ 0 & 27 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -8 & 0 \\ 70 & 27 \end{bmatrix} \end{aligned}$$